SINGULARITIES IN UNIVERSES WITH NEGATIVE COSMOLOGICAL CONSTANT

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SUMMARY

It is well-known that many universes with negative cosmological constant contain singularities. We shall generalize this result by proving that all closed universes with negative cosmological constant are both future and past timelike geodesically incomplete if the strong energy condition holds. No global causality conditions or restrictions on the initial data are used in the proof. Futhermore, we shall show that all open universes with a Cauchy surface and a negative cosmological constant are singular if the strong energy condition holds.

I. INTRODUCTION

There has been a tendency in recent years to treat the cosmological constant not as an arbitrary constant appearing on the "geometrical" side of the Einstein field equations, but rather as part of the stress-energy tensor - as part of a term on the "physical" side of Einstein's equations. Zel'dovich, for example, has argued (1967, 1968; Zel'dovich and Novikov, 1971) that quantum fluctuations can give rise to a non-zero vacuum expectation value for the stress-energy tensor. Furthermore, he contends that this vacuum expectation value must have the form Λg_{ab}, where Λ is a constant. As emphasized by Gunn and Tinsley (1975), this sort of treatment of the cosmological constant has a great aesthetic advantage over the more traditional view, for the cosmological constant is no longer an arbitrary constant to be determined by astronomical observations. It is instead a number which can in principle be calculated from the results of local measurements on quantized fields.

Several attempts have been made to calculate this number (Zel'dovich and Novikov, 1971; Streeruwitz, 1975; Dreitlein, 1974; see also Ford, 1975). For our purposes it will not be necessary to know its exact value: we need only know its sign. We shall show in this paper that if the cosmological constant is negative - and one of the above authors (Dreitlein, 1974; see however Weinberg, 1976) does indeed obtain a cosmological constant with this sign - then the universe must have singularities both in the past and in the future of an initial spacelike hypersurface. This result is completely independent of the boundary conditions imposed on the initial hypersurface. It depends only on very weak global assumptions which are used in virtually every calculation of the future behavior of the universe.

Previous proofs of the occurrence of singularities have required that knowledge of the cosmological constant be supplemented with boundary conditions, such as the assumption of the Universe's exact homogeneity and isotropy (Misner, Thorne, and Wheeler, 1973). Even the sungularity theorems of Hawking and Penrose require some information about the initial conditions: for example, it must be shown that the universe contains either a closed trapped surface of a point p such that on every past null geodesic from p the divergence θ of the null goedesics from p become negative (Hawking and Ellis, 1973, p. 267). Local observations are insufficient to determine if these conditions hold. Astronomical observations must be used. We shall show in this paper that in one case - the actual case according to Dreitlein (1974) - these astronomical observations are unnecessary; we can determine certain aspects of the universe's future (and past) behavior from laboratory measurements alone if these measurements imply a negative cosmological constant.

In section II, we shall show that all closed universes with negative cosmological constant have singularities in the form of incomplete time-like geodesics both in the past and in the future of the compact spacelike hypersurface. The only global assumption needed in the proof is the strong energy condition; no global causality assumption is necessary. Thus, the theorem proved in this section is the solution to a problem posed by Geroch (1970, p. 266,288): Prove that when $\Lambda < 0$ there does not exist even one non-singular closed universe solution to Einstein's equations. (By a "solution to Einstein's equations" Geroch means a spacetime whose stress-energy tensor satisfies the strong energy condition (Geroch 1970; p. 264).[1]

[1] This is a very reasonable use of the word "solution," for if no restriction were made on the stress-energy tensor, then any spacetime would be a "solution"

to Einstein's equations; Einstein's equations would be vacuous. Futhermore, the strong energy condition is obeyed by all known forms of matter (Hawking and Ellis, 1973, p. 95).

In Section III, we shall show that all open universes with negative cosmological constant and a Cauchy surface S have singularities in the form of incomplete timelike goedesics both in the past and in the future of S. The theorem proved in this section requires a global causality condition in addition to the strong energy condition: it is necessary to assume the existence of a Cauchy surface S. (The existence of a Cauchy surface S means that it is possible to predict the future (and past) behavior of the universe using initial data available on S.) However, the existence of a Cauchy surface makes it possible to prove that not only do singularities occur, but also that no timelike curve has a proper time length greater than $2\pi(3/|\Lambda|)^{1/2}$ - and this result is true for any universe, open or closed, with a Cauchy surface.

The notation, sign conventions, etc., used in this paper are the same as those used in Hawking and Ellis (1973), (hereafter referred to as HE), unless otherwise noted. For example, Latin indices take the values 1, 2, 3, 4 while Greek indices take the values 1, 2, 3. The metric signature is + 2.

II. CLOSED UNIVERSES

A closed universe is a spacetime (M,g) which contains a compact spacelike three-surface S without edge (Geroch, 1970, pp. 265, 267; HE, p. 273). The existence of singularities in closed universes with negative cosmological constant is established in the following theorem.

Theorem I: Let (M,g) be a closed universe on which the Einstein field equations hold. Then (M,g) is timelike goedesically incomplete if:

(1) $(T_{ab} - \frac{1}{2} g_{ab} T) U^a U^b \geq 0$ for all timelike vectors U^a. (i.e., the strong energy condition holds.)

(2) $\Lambda < 0$

The proof of Theorem I will require the following

Lemma: Let S be a spacelike three-surface in a spacetime (M,g) on which the Einstein equations and the strong energy condition hold. Let $\gamma(t)$ be a timelike geodesic which intersects S orthogonally, where t measures proper time along the geodesic, and $t = 0$ at S. Then if $\Lambda < 0$ on (M,g), there will be a point conjugate to S along $\gamma(t)$ within a proper time distance $\pi(3/|\Lambda|)^{1/2}$ from S, provided that $\gamma(t)$ can be extended that far.

Proof: Roughly speaking, a point p is said to be conjugate to S along $\gamma(t)$ if a geodesic which is close to $\gamma(t)$ and normal to S intersects $\gamma(t)$ at p. To make this precise, we recall from HE (pp. 96-100) that a point p is said to be conjugate to S along $\gamma(t)$ if the expansion θ along $\gamma(t)$ becomes infinite at p and satisfies

$$\theta = \chi^a{}_a \tag{II.1}$$

at S, where χ_{ab} is the second fundamental form of S. The expansion θ satisfies the equation

$$\frac{d\theta}{dt} = - R_{ab} V^a V^b - 2\sigma^2 - \frac{1}{3}\theta^2 \tag{II.2}$$

where V^a is the unit tangent vector to $\gamma(t)$ and $\sigma^2 \geq 0$. In addition, it can be shown (HE, p. 97, 100) that p is conjugate to S along $\gamma(t)$ if and only if a function y, defined by $\theta \equiv (1/y)dy/dt$, satisfies $y = 0$ at p. If we define a new function x by the relation $x^3 \equiv y$, then $\theta = (3/x)dx/dt$, and (II.2) becomes

$$\frac{d^2x}{dt^2} + F(t)\, x = 0 \tag{II.3}$$

where

$$F(t) \equiv \frac{1}{3}(R_{ab}V^aV^b + 2\sigma^2) \tag{II.4}$$

Since $x^3 = y$, y will be zero at p if and only if $x = 0$ at p. Thus, there will be a point conjugate to S along $\gamma(t)$ within the proper time interval [0,b] if <u>every</u> solution to (II.3) has at least one zero in that interval. (This is a sufficient condition for a point conjugate to S, not a necessary condition, since we are really concerned only with those solutions to (II.3) which satisfy (II.1).)

It can be shown (Hille, 1969) that a sufficient condition for every solution for every solution to (II.3) to have at least one zero in [0,b] is

$$\inf_{0 \leq t \leq b} F(t) \geq (\pi/b)^2 \tag{II.5}$$

The Einstein equations can be written (HE, p. 74)

$$R_{ab} = 8\pi(T_{ab} - \frac{1}{2} g_{ab} T) + \Lambda g_{ab}$$

which gives

$$\inf_{0\le t\le b} F(t) = \inf_{0\le t\le b} \frac{1}{3}\left[8\pi (T_{ab} - \frac{1}{2} g_{ab} T) V^a V^b - \Lambda + 2\sigma^2 \right]$$

$$\ge \frac{1}{3} |\Lambda|$$

using $\Lambda < 0$ and the strong energy condition.

Thus, there will be a point conjugate to S along $\gamma(t)$ within a proper time distance $\pi(3/|\Lambda|)^{1/2}$ from S, provided $\gamma(t)$ can be extended that far.

Proof of Theorem I: (the proof is a modification of the proof of Theorem 4 in HE, p. 273). It can be shown (HE, pp. 204-205) that there exists a covering manifold $\hat{M}$ to M such that each connected component of the image of S is diffeomorphic to S and is a partial Cauchy surface in $\hat{M}$. If there are incomplete goedesics in $\hat{M}$, then there will be incomplete geodesics in M. Therefore, the proof can be carried out in $\hat{M}$; $\hat{S}$ will denote one connected component of the image of S.

IF M (and hence $\hat{M}$) were timelike goedesically complete, there would be a point conjugate to $\hat{S}$ on every future-directed goedesic orthogonal to $\hat{S}$ within a proper time distance $b = \pi(3/|\Lambda|)^{1/2}$ from $\hat{S}$. But to every point $q \in D^+(\hat{S})$ there is a future-directed geodesic orthogonal to $\hat{S}$ which does not contain any point conjugate to $\hat{S}$ between $\hat{S}$ and q. (HE,p. 217). Let $\beta: \hat{S} \times [0,b] \to \hat{M}$ be the differentiable map which takes a point $p \in \hat{S}$ a proper time distance $t \in [0,b]$ along the future-directed geodesic through

p orthogonal to $\hat{S}$. Then $\beta(\hat{S} \times [0,b])$ would be compact and would contain $\overline{D^+(\hat{S})}$. Since the intersection of a compact set and a closed set is compact, this implies that $\overline{D^+(\hat{S})}$ and hence $\overline{H^+(\hat{S})}$ would be compact. $(H^+(\hat{S}) = \overline{H^+(\hat{S})} \subset \overline{D^+(\hat{S})})$.

Consider now a point $q \in H^+(\hat{S})$. The function $d(\hat{S},q)$ would be less than or equal to $b = \pi(3/|\Lambda|)^{1/2}$ since every past-directed non-spacelike curve from q to $\hat{S}$ would consist of a (possibly zero) null geodesic segment in $H^+(\hat{S})$ followed by a non-spacelike curve in $D^+(\hat{S})$. (See HE, p. 215 for the definition of $d(\hat{S},q)$.) Since d is lower semi-continuous, there would exist an infinite sequence of points $r_n \in D^+(\hat{S})$ converging to q such that $d(\hat{S},r_n)$ converged to $d(\hat{S},q)$. There would correspond to each r_n at least one element $\beta^{-1}(r_n)$ of $\hat{S} \times [0,b]$. Furthermore, there would be an element $\beta(p,t)$ which would be a limit point of the $\beta^{-1}(r_n)$ since $\hat{S} \times [0,b]$ is compact. By continuity we would have $t = d(\hat{S},q)$ and $\beta(p,t) = q$. Hence to every point $q \in H^+(\hat{S})$ there would be a timelike geodesic of length $d(\hat{S},q)$ from $\hat{S}$. Now let $q_1 \in H^+(S)$ be a point to the past of q on the same null geodesic generator λ of $H^+(S)$. If we were to join the geodesic of length $d(\hat{S},q_1)$ from $\hat{S}$ to q_1 to the segment of λ between q_1 and q, we would obtain a non-spacelike curve of length $d(\hat{S},q_1)$ from $\hat{S}$ to q which could be varied to give a longer curve between these endpoints. (HE, p. 112). Thus the function $d(\hat{S},q)$ with $q \in H^+(\hat{S})$, would strictly decrease along every past-directed generator of $H^+(\hat{S})$. Now these generators have no past endpoints. (HE, p. 203). But this contradicts the fact that $d(\hat{S},q)$ $q \in H^+(\hat{S})$, would have a minimum on the compact set $H^+(\hat{S})$ since $d(\hat{S},q)$ is lower semi-continuous in q. Thus some future-directed timelike geodesic orthogonal to $\hat{S}$ must be incomplete. Furthermore, this incomplete geodesic must have to the future of $\hat{S}$ a proper time length less than or equal to $\pi(3/|\Lambda|)^{1/2}$. A similar argument with the past-directed geodesics orthogonal to $\hat{S}$ will show that there is at least one timelike geodesic from $\hat{S}$ which is incomplete in the past direction.

In the proof of Theorem I we have also proven the following:

<u>Corollary</u>: There exists at least one timelike geodesic orthogonal to S which is incomplete to the future of S, and at least one timelike geodesic orthogonal to S which is incomplete to the past of S. Furthermore, the proper time length from S of these incomplete geodesics is less than or equal to $\pi(3/|\Lambda|)^{1/2}$.

III. OPEN UNIVERSES

We shall follow Geroch (1970, p. 267) and define an <u>open universe</u> to be a spacetime which does <u>not</u> contain a compact three-surface without edge; that is, an open universe is any spacetime which is not a closed universe. The following theorem establishes the existence of singularities in a wide class of open universes with negative cosmological constant: the open universes in which it is possible to predict the future.

<u>Theorem II</u>: Let (M,g) be a spacetime on which the Einstein field equations hold. Then (M,g) is timelike geodesically incomplete if:

(1) The strong energy condition holds

(2) $\Lambda < 0$

(3) There exists a Cauchy surface S in M.

<u>Proof</u>: From the lemma proved in the previous section and the fact that to each point $q \in D^+(S)$ there is a future-directed timelike geodesic orthogonal to S of proper time length $d(S,q)$ which does not contain any point conjugate to S between S and q, it follows that there is in $D^+(S)$ no future-directed timelike curve from S with proper time length greater than $\pi(3/|\Lambda|)^{1/2}$. However, all future-directed timelike curves from S remain in $D^+(S)$ since S is a Cauchy surface. Furthermore, all timelike curves intersect S. Thus, <u>all</u> timelike goedesics are incomplete in the future direction, and their lengths from S are less than or equal to $\pi(3/|\Lambda|)^{1/2}$. A similar result holds for the past direction. Since the maximum proper time distance from S in either direction is $\pi(3/|\Lambda|)^{1/2}$, no timelike curve has a length greater than $2\pi(3/|\Lambda|)^{1/2}$.

We have also proven:

Corollary: All timelike geodesics are both future and past incomplete, and no timelike curve has a proper time length greater than $2\,\pi(3/|\Lambda|)^{1/2}$.

Note that Theorem II and its Corollary apply to all universes, both open and closed, which have a Cauchy surface and a negative cosmological constant (and which satisfy the strong energy condition and the Einstein equations). However, condition (3) (the existence of a Cauchy surface) is a necessary condition in Theorem II only in the open universe case. And it is indeed necessary in the open universe case. Anti-de Sitter space (HE, pp. 131-134) is an example of an open universe which satisfies all the conditions of Theorem II except condition (3), and anti - de Sitter space is geodesically complete (Penrose, 1968). Furthermore, anti-de Sitter space is stably causal (i.e., it is possible to vary the metric slightly without getting closed timelike curves), so condition (3) cannot be replaced by a weaker causality condition such as the strong causality condition or the chronology condition.

However, condition (3) is a very reasonable assumption to make from the physical point of view,[2] for the existence of a Cauchy surface is

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[2] Anti - de Sitter space is physically unrealistic, for $T_{ab} = C_{abcd} = 0$.

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equivalent to the assumption that the universe is deterministic. Thus, Theorem II can be interpreted as saying that if local experiments imply a negative cosmological constant, then either singularities occur in the future, or the final state of the universe is unpredictable - and the latter would also be a rather singular occurrence!

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SUMMARY

It is well-known that many universes with negative cosmological constant contain singularities. We shall generalize this result by proving that <u>all</u> closed universes with negative cosmological constant are both future and past timelike geodesically incomplete if the strong energy condition holds. No global causality conditions or restrictions on the initial data are used in the proof. Futhermore, we shall show that all open universes with a Cauchy surface and a negative cosmological constant are singular if the strong energy condition holds.

Biographical Sketch

I was born and raised in Andalusia, a small town in southern Alabama. My interest in physics dates back to my kindergarten days (circa 1952) when I became fascinated with von Braun's visions of interplanetary flight. By the time I entered M.I.T. as an 18 year old freshman in 1965, however, this interest had metamorphosed into interest in fundamental physics, with particular attention to the role of Time in scientific theories. Graduating from M.I.T. in 1969, I became a graduate student at the University of Maryland, where I am now working toward a Ph.D. in General Relativity with Dieter Brill as thesis supervisor.

My outside interests include hiking, reading Russian literature and science fiction, and studying history and philosophy.

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