CAUSALLY SYMMETRIC SPACETIMES

Abstract

Causally symmetric spacetimes are spacetimes with $J^+(S)$ isometric to $J^-(S)$ for some set S. We discuss certain properties of these spacetimes, showing for example that if S is a maximal Cauchy surface with matter everywhere on S, then the spacetime has singularities in both $J^+(S)$ and $J^-(S)$. We also consider totally vicious spacetimes, a class of causally symmetric spacetimes for which $I^+(p) = I^-(p) = M$ for any point p in M. Two different notions of stability in General Relativity are discussed, using various types of causally symmetric spacetimes as starting points for perturbations.

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1. Introduction

The concept of symmetry is basic to physics. Symmetry in General Relativity is usually based on a local one-parameter group of isometries generated by a vector field: a Killing vector field. In this paper I shall develop a notion of symmetry which is based on the global causal structure of spacetime. The reasons for analyzing spacetimes with causal symmetries are in part the same as the reasons for considering spacetimes with Killing symmetries: first, the spacetimes having such symmetries mimic important known features of the actual Universe while simplifying the problem of solving the field equations; and second, the mathematical simplicity of such spacetimes allows them to be used as easily understood examples of exotic spacetime structures - structures which may form a part of the actual Universe.[1]

I shall discuss both applications of symmetries in this paper. After setting down the definitions and basic relations between the various types of causal symmetries in Section 2, I shall show in Section 3 that all time symmetric universes which contain matter everywhere are singularity symmetric. That is, these universes have singularities both to the past and to the future of the spacelike hypersurface about which the universe is time symmetric. Many writers believe[2,3,4] that the actual Universe is closed, and the evidence suggests[5] that it is isotropic and homogeneous. This implies the existence of a surface of time symmetry. Thus if we assume that this surface of time symmetry is a Cauchy surface, then it follows that the Universe can exist for only a finite time. In general, Section 3 will be devoted to a discussion of the conditions which must be imposed on a spacetime in order to make it singularity symmetric.

In Section 4 I shall briefly discuss some of the properties of totally vicious spacetimes, the class of causally symmetric spacetimes for which $I^+(p) = I^-(p) = M$, the entire spacetime, for any point p in M. These spacetimes provide a counter-example to a theorem by Hawking and Sachs: A causally simple spacetime is stably causal.

Causally symmetric spacetimes have one advantage over Killing symmetric spacetimes. Causal symmetries are quite amendable to analysis by the global techniques developed by Hawking and Penrose, and using these methods it is easy to prove that many of the properties of these spacetimes are "stable". I have placed "stable" in quotes because there are two notions of stability used in General Relativity. First, a spacetime property is said to be stable if it still occurs when the initial data is perturbed. Second, a property is said to be stable if it persists when the metric is changed slightly at every point in the spacetime. I shall make these different notions of stability more precise in Section 5, showing that the singularity symmetry of some of the spacetimes considered in Section 3 is a stable property in the first sense, and that the total viciousness of the spacetimes of Section 4 is a stable property in the second sense.

The notation of this paper is the same as that of Hawking and Ellis[6], hereafter denoted HE. I shall assume that the cosmological constant is zero.

2. Definitions and Basic Relationships

The three basic causal sets, $J^+(S)$, $I^+(S)$, and $D^+(S)$ give rise to the following three definitions of symmetry:

Definition: A spacetme (M,g) will be called causally symmetric about a set S it $J^+(S)$ is isometric to $J^-(S)$, written

$$J^+(S) \simeq J^-(S)$$

Definition: A spacetime (M,g) will be called chronologically symmetric about a set S if

$$I^+(S) \simeq I^-(S)$$

Definition: A spacetime (M,g) will be called Cauchy symmetric about a set S if

$$D^+(S) \simeq D^-(S)$$

Since all spacetimes have the above symmetries if S = M, some restriction will have to be placed on S for the definitions to be useful. In Section 3, we will require S to be a partial Cauchy surface, and on this structure we can define another notion of "Cauchy" symmetry:

Definition: A spacetime (M,g) is called time symmetric if there exists a partial Cauchy surface at each point of which the extrinsic curvature χ_{ab} vanishes.

This is the definition of time symmetry as given by Misner, Thorne, and Wheeler[7]. There are other definitions of time symmetry in the literature[8]. For example, in Harrison, Thorne, Wakano, and Wheeler we find the following definition of time symmetry: "A spacelike hypersurface is said to be a hypersurface of time symmetry when the dynamical history and the 4-geometry on the future side of this hypersurface is the time-reversed image of the dynamical history and the 4-geometry in the past"[9]. As I interpret this statement, the above authors claim that a spacelike hypersurface S is a surface of time symmetry if the spacetime is both Cauchy symmetric and causally symmetric about S. (The time reversal of the dynamical history gives rise to the Cauchy symmetry and the time reversal of the 4-geometry gives rise to the causal symmetry.) Note, however, that neither $D^+(S) \simeq D^-(S)$ nor $J^+(S) \simeq J^-(S)$ imply $\chi_{ab} = 0$. For example, let S have the topology R^3 and let (x,y,z) be a Euclidean coordinate system on S, with initial data set (h_{ab}, χ_{ab}) on S. (h_{ab} is the metric on S) Then if $h_{ab}(x,y,z) = h_{ab}(x,y,-z)$ and $\chi_{ab}(x,y,z) = -\chi_{ab}(x,y,-z)$ with $\chi_{ab} \neq 0$ except at points for which z = 0, we can evolve this data so that $D^+(S) \simeq D^-(S)$ and $J^+(S) \simeq J^-(S)$. In other words, this initial data set is <u>globally</u> time symmetric (i.e., $J^+(S) \simeq J^-(S)$ and $D^+(S) \simeq D^-(S)$) but not <u>locally</u> time symmetric (i.e., for any point p in S with $z \neq 0$, we have $J^+(p)$ <u>not</u> isometric to $J^-(p)$.).

Furthermore, $\chi_{ab} = 0$ on a partial Cauchy surface S does not imply any of the three causal symmetries. For example, remove the point (x = y = z = 0, t = +1) from Minkowski space. In the resulting spacetime the hypersurface t = 0 has $\chi_{ab} = 0$, but $D^+(S)$ is not isometric to $D^-(S)$. We do, however, have the following:

<u>Proposition 1</u>: If the spacetime (M,g) is time symmetric about S and if $D(S) \equiv D^+(S) \cup D^-(S)$ is the maximal Cauchy development from S, then $D^+(S) \simeq D^-(S)$.

This result follows immediately from the existence and uniqueness of the maximal development from S, proven in Chapter 7 of HE. In a similar manner, we prove

<u>Propostion 2</u>: If S is a time symmetric Cauchy surface, then S is causally symmetric about S.

In the next section we will show that singularities develop both to the past and to the future of a maximal hypersurface[10] S provided there is matter present everywhere on S. Intuitively, the notion of "matter present everywhere on S" means " the energy density is non-zero at each point of S." We can make this intuitive notion precise via one of the following conditions:

Definition: The weak ubiquitous energy condition is said to hold on a set S if $T_{ab} V^a V^b > 0$ for all timelike or null vectors V^a at each point p in S.

Definition: The strong ubiquitous energy condition is said to hold on a set S if $(T_{ab} - \frac{1}{2} g_{ab} T) V^a V^b > 0$ for all timelike or null vectors V^a at each point p in S.

All observed matter fields obey both of the above conditions at a point p if $T^{ab} \neq 0$ at p. However, there are certain fields which are often used as approximations to actual fields that do not satisfy one or both of the above conditions if $T_{ab} \neq 0$. For example, a null fluid moving entirely in the V^a direction would give $T_{ab} V^a V^b = 0$ with $T_{ab} \neq 0$. Furthermore, a massive scalar field could violate the strong ubiquitous energy condition while satisfying the weak ubiquitous energy condition. (see page 95 of HE) Since it is unlikely that the matter at a given point would consist entirely of radiation moving in one direction, and since a massive scalar field with $T^{ab} \neq 0$ could violate the strong ubiquitous energy condition only at such extremely high densities that we cannot trust the matter equations, it is reasonable to assume that the above energy conditions hold at a point p whenever $T_{ab} \neq 0$ at p.

The condition $T_{ab} \neq 0$ for all $p \in M$ was apparently originally proposed by Aristotle (Nature abhors a vacuum), and later defended by numerous authors, among them G. W. Leibniz, who supported it with an argument which is cogent even in the world-view of General Relativity: at any point in spacetime we expect there will be a little randomly oriented radiation present, even in what would otherwise be a perfect vacuum. The microwave background radiation for example, is expected to be present everywhere in spacetime, except perhaps where there is matter to shield it out. This random background radiation would be sufficient to satisfy both of the ubiquitous energy conditions; even in radiation shielded regions there would be quantum mechanical zero-point radiation which would in itself be sufficient to satisfy the condition. Thus the above ubiquitous energy conditions seem to be eminently reasonable conditions to impose on the whole of spacetime, though for our purposes we will need to impose them only on an initial spacelike hypersurface.

3. Singulatity Symmetric Spacetimes

We will now show that any spacetime which is time symmetric about a spacelike hypersurface S (or more generally, for which S is a maximal hypersurface) and which has matter everywhere on S has singularities both to the future and to the past of S. The first two theorems will apply to the case in which S is compact, and they require no global causality assumption. The third theorem, which handles the non-compact case, will require a causality assumption: S is required to be a Cauchy surface. The first theorem is really a special case of the second. It is included separately for two reasons. First of all, it facilitates comparison with a similar theorem by Brill and Flaherty[11], and second, since its conclusions depend explicitly on the initial data and not on a more general global generic condition, it will be used to prove the stability of a class of singularity symmetric spacetimes.

Theorem 1: Suppose that a spacetime (M,g) contains a maximal spacelike hypersurface S which is compact and edgeless. Then there is at least one timelike geodesic which is incomplete to the future of S, and at least one timelike geodesic which is incomplete to the past of S, provided:

(1) The Einstein equations hold on (M,g);

(2) The strong energy condition holds on (M,g);

(3) The strong ubiquitous energy condition holds on S.

Theorem 2: Suppose that a spacetime (M,g) contains a maximal spacelike hypersurface S which is compact and edgeless. Then there is at least one timelike geodesic which is incomplete to the future of S, and at least one timelike geodesic which is incomplete to past of S, provided:

(1) The Einstein equations hold on (M,g);

(2) The strong energy condition holds on (M,g);

(3) For every timelike geodesic γ with $\gamma \cap S \neq \phi$, there are points

p, q in $J^+(S)$ and $J^-(S)$ respectively such that at p and q, $V^a V^b V_{[c} R_{d]ab[e} V_{f]} \neq 0$, where V^a is the unit tangent vector to γ.

Proof: Clearly Theorem 1 is a special case of Theorem 2, for let p = q be a point in $\gamma \cap S$. Then $R_{ab} V^a V^b > 0$ at p = q by conditions (1) and (2) of Theorem 1. But this implies $V^a V^b V_{[c} R_{d]ab[e} V_{f]} \neq 0$ at p = q (see page 540 of reference 12), so condition (3) of Theorem 2 holds. Thus we need only prove Theorem 2. (The proof is a modification of the proof of Theorem 4 in HE, p.273.) It can be shown (HE, pp. 204-205) that there exists a covering manifold $\hat{M}$ to M such that each connected component of the image of S is diffeomorphic to S and is a partial Cauchy surface in $\hat{M}$. If there are incomplete timelike geodesics both to the future and to the past of any one connected component $\hat{S}$ of the image of S, then there will be incomplete timelike geodesics both to the future and to the past of S in M. Therefore, the proof can be carried out in $\hat{M}$. We first show that any timelike geodesic γ which intersects $\hat{S}$ orthogonally will have a point conjugate to $\hat{S}$ both to the future and to the past of $\hat{S}$, provided γ can be extended that far. Recall that a point p on γ is said to be conjugate to $\hat{S}$ along γ if there is a Jacobi field along γ which is not identically zero but vanishes at p and satisfies the initial condition

$$V_{a;b} = \chi_{ab} \qquad (3\text{-}1)$$

at $\hat{S}$ (HE, pp. 96-100). The Jacobi fields along $\gamma(t)$ which satisfy the above initial condition can be written (HE, p. 99):

$$Z^\alpha = A_{\alpha\beta}(t) \, Z^\beta \Big|_q$$

where $\alpha,\beta = (1,2,3)$, t is the proper time along $\gamma(t)$ [with $t = 0$ at q], and q is the point at which γ intersects $\hat{S}$. At q, $A_{\alpha\beta}$ is the unit matrix, and the point p will be conjugate to $\hat{S}$ along $\gamma(t)$ if and only if the determinant of $A_{\alpha\beta}$ vanishes at p. If we define

$$x^3 \equiv \det (A_{\alpha\beta})$$

$$\theta = \frac{3}{x}\frac{dx}{dt} \qquad (3\text{-}2)$$

$$\sigma_{\alpha\beta} = A^{-1}{}_{\gamma(\beta}\frac{d}{dt}A_{\alpha)\gamma} - \frac{1}{3}\delta_{\alpha\beta}\,\theta$$

then it can be shown (HE, pp. 96-101) that $A_{\alpha\beta}$ satisfies

$$\frac{d\theta}{dt} = - R_{ab}V^a V^b - 2\sigma^2 - \frac{1}{3}\theta^2 \qquad (3 - 3)$$

where $2\sigma^2 = \sigma_{\alpha\beta}\sigma^{\alpha\beta} \geq 0$. Using $\theta = (3/x)dx/dt$, (3-3) can be written

$$\frac{d^2x}{dt^2} + F(t)x = 0 \qquad (3\text{-}4)$$

where

$$F(t) = \frac{1}{3}(R_{ab}V^a V^b + 2\sigma^2) \qquad (3\text{-}5)$$

Since $x^3 = \det (A_{\alpha\beta})$, $\det (A_{\alpha\beta})$ will be zero at p if and only if $x = 0$ at p. At q, we have (HE, p. 100):

$$\theta = V^a{}_{;a} = \chi^a{}_a = \frac{3}{x}\frac{dx}{dt} = 0$$

Thus, showing that any future-complete timelike geodesic $\gamma(t)$ orthogonal to $\hat{S}$ has a point conjugate to $\hat{S}$ to the future of $\hat{S}$ is equivalent to showing that the solution to (3-4) which satisfies the initial conditions

$$x = 1 \quad , \quad \frac{dx}{dt} = 0 \tag{3-6}$$

at $t = 0$ has a zero in $(0, + \infty)$.

By conditions (1) and (2), $F(t) \geq 0$ in $[0, + \infty)$, and condition (3) implies that there exists a value t_1 in $[0, + \infty)$ for which $F(t) > 0$. Thus, from equation (3-4), we have a value t_2 for which

$$\left.\frac{dx}{dt}\right|_{t_2} = - \int_0^{t_2} F(t)\, x(t)\, dt \quad < \quad 0 \tag{3-7}$$

Since $F(t) \geq 0$, this means implies a zero of x for some t in $[0, + \infty)$. A similar argument shows that any past-complete timelike geodesic $\gamma(t)$ orthogonal to $\hat{S}$ has a point conjugate to $\hat{S}$ to the past of $\hat{S}$.

By Proposition 7.24 of Penrose[13], the location of the first conjugate point to $\hat{S}$ on γ varies continuously with the point at which γ intersects $\hat{S}$ and γ. Thus the proper time length to the first conjugate point of $\hat{S}$ along the future-directed timelike geodesics orthogonal to $\hat{S}$ is a continuous function defined on $\hat{S}$, provided all γ are future-complete. Thus it attains its maximum value b on the compact set $\hat{S}$: if $\hat{M}$ were timelike geodesically complete to the future of $\hat{S}$, there would be a point conjugate to $\hat{S}$ on every future-directed geodesic orthogonal to $\hat{S}$ within a proper time distance b. But to every point $q \in D^+(\hat{S})$ there is a future-directed geodesic orthogonal to $\hat{S}$ which does not contain any point conjugate to $\hat{S}$ between $\hat{S}$ and q (HE, p. 217). Let $\beta : \hat{S} \times [0,b] \to M$ be the differential map which takes a point $p \in \hat{S}$ a proper time distance $t \in [0,b]$ along the future-directed geodesic through p orthogonal to $\hat{S}$. Then $\beta(\hat{S} \times [0,b])$ would be compact and would contain $\overline{D^+(\hat{S})}$. Since the intersection of a compact set and a closed set is compact, this implies that $\overline{D^+(\hat{S})}$ and hence $H^+(\hat{S})$ would be compact.

Consider now a point $q \in H^+(\hat{S})$. The function $d(\hat{S},q)$ would be less than or equal to b, since every past-directed non-spacelike curve from q to $\hat{S}$ would consist of a (possibly zero) null geodesic segment in $H^+(\hat{S})$ followed by a non-spacelike curve in $D^+(\hat{S})$. (see HE, p. 215 for the definition of $d(\hat{S},q)$.) Since d is lower semi-continuous, there would exist an infinite sequence of points $r_n \in D^+(\hat{S})$ converging to q such that $d(\hat{S},r_n)$ converged to $d(\hat{S},q)$. There would correspond to each r_n at least one element $\beta^{-1}(r_n)$ of $\hat{S} \times [0,b]$. Furthermore, there would be an element $\beta^{-1}(p,t)$ which would be a limit point of the $\beta^{-1}(r_n)$ since $\hat{S} \times [0,b]$ is compact. By continuity we would have $t = d(\hat{S},q)$ and $\beta(p,t) = q$. Hence to every point $q \in H^+(\hat{S})$ there would be a timelike geodesic of length $d(\hat{S},q)$ from $\hat{S}$. Now let $q_1 \in H^+(\hat{S})$ be a point to the past of q on the same null geodesic generator λ of $H^+(\hat{S})$. If we were to join the geodesic of length $d(\hat{S},q_1)$ from $\hat{S}$ to q_1 to the segment of λ between q_1 and q, we would obtain a non-spacelike curve of length $d(\hat{S},q_1)$ from $\hat{S}$ to q which could be varied to give a longer curve between these endpoints. (HE, p. 112). Thus the function $d(\hat{S},q)$, with $q \in H^+(\hat{S})$, would strictly decrease along every past-directed generator of $H^+(\hat{S})$. Now these generators have no past endpoints. (HE, p. 203). But this contradicts the fact that $d(\hat{S},q)$, $q \in H^+(\hat{S})$, would have a minimum on the compact set $H^+(\hat{S})$ since $d(\hat{S},q)$ is lower semi- continuous in q. Thus some future-directed timelike geodesic orthogonal to $\hat{S}$ must be incomplete. A similar argument with the past-directed geodesics orthogonal to $\hat{S}$ will show that there is at least one timelike geodesic from $\hat{S}$ which is incomplete in the past direction. Δ

Condition (3) of Theorem 2 is a very weak condition to impose on a spacetime. We can have $V^a V^b V_{[c} R_{d]ab[e} V_{f]} = 0$ along $\gamma \cap J^+(S)$ only if $R_{ab}V^aV^b$ vanishes at every point of $\gamma \cap J^+(S)$, and then only if the Weyl tensor is related in a very particular way to γ ($C_{abcd}V^bV^c = 0$) at <u>every</u> point of $\gamma \cap J^+(S)$. Hawking and Penrose have pointed out[12] that for any physically realistic spacetime, this would not even occur at <u>any</u> point of <u>any</u> γ!

In order to prove singularity symmetry about a maximal <u>non-compact</u> spacelike hypersurface, we will need to impose stronger conditions on the spacetime than were necessary in the compact case. We shall need a causality assumption - S will be assumed to be a Cauchy surface - and we shall need to assume that the matter density is bounded away from zero for some finite proper time for all observers which travel on geodesics hitting S orthogonally. A stronger initial condition on the matter than that imposed in the compact case is a necessary condition for singularity symmetry: there are spacetimes for which $T^{ab}V_aV_b > 0$ <u>everywhere</u> on a maximal Cauchy surface S and which is singularity-free.An example would be a static, spherical star which carries an electric charge. Thus the following theorem will not apply to asymptotically flat spacetimes. However, we would expect its condition (3) to hold for a spacetime for which the matter density is roughly constant on S.

<u>Theorem 3</u>: Suppose that (M,g) contains a maximal Cauchy surface S. Then there is at least one timelike geodesic which is incomplete to the future of S, and at least one timelike geodesic which is incomplete to the past of S,

provided:

(1) The Einstein equations hold on (M,g);

(2) The strong energy condition holds on (M,g);

(3) There exist positive constants a, b such that

$$\left| \int_0^a (T_{ab} - \tfrac{1}{2} g_{ab} T) V^a V^b \, dt \right| \geq b$$

for <u>every</u> timelike geodesic segment $\gamma \cap J^+(S)$ <u>and</u> $\gamma \cap J^-(S)$, where γ is a geodesic intersecting S orthogonally and the proper time t along γ is zero at S.

<u>Proof</u>: We first show that every future-directed timelike geodesic γ orthogonal to S has a conjugate point to S within a proper time distance $(a + 3/8\pi b)$. Suppose not. Then we have $x > 0$ in this interval and

$$\left.\frac{dx}{dt}\right|_{t=a} = - \int_0^a F(t)\, x(t)\, dt = - \frac{8\pi}{3} \int_0^a [(T_{ab} - \tfrac{1}{2} g_{ab} T) V^a V^b + \frac{2\sigma^2}{8\pi}] x \, dt$$

$$\leq - \frac{8\pi}{3} x(a) \int_0^a (T_{ab} - \tfrac{1}{2} g_{ab} T) V^a V^b \, dt \leq - \frac{8\pi}{3} x(a)\, b$$

Since $dx/dt \leq dx/dt|_{t=a}$ for all $t \geq a$ before the first zero of x, there must be a zero of x within a distance c of $t = a$, where c is defined by

$$\left.\frac{dx}{dt}\right|_{t=a} = - \frac{x(a)}{c}$$

Thus

$$c = - \left.\frac{x(a)}{dx/dt}\right|_{t=a} \leq - \frac{x(a)}{-\frac{8\pi}{3} x(a)\, b} = \frac{3}{8\pi b}$$

Hence a zero of x occurs within a distance $(a + 3/8\pi b)$ from S, and this means a point conjugate to S along γ.

From this result and the fact that to each point $q \in D^+(S)$ there is a future-directed timelike geodesic orthogonal to S of proper time length $d(S,q)$ which does not contain any point conjugate to S between S and q, it follows that there is in $D^+(S)$ no future-directed timelike curve from S with proper time length greater than $(a + 3/8\pi b)$. However, all future-directed timelike curves from S remain in $D^+(S)$ since S is a Cauchy surface. Furthermore, all timelike curves intersect S. Thus, all timelike geodesics are incomplete in the future direction, and their lengths from S are less than or equal to $(a + 3/8\pi b)$. A similar result holds for the past direction. Since the maximum proper time distance from S in either time direction is $(a + 3/8\pi b)$, no timelike curve has a length greater than $2(a + 3/8\pi b)$.

We have also proven:

Corollary: All timelike geodesics are both future and past incomplete, and no timelike curve has a proper time length greater than $2(a + 3/8\pi b)$.

Note that Theorem 3 and its Corollary apply to all spacetimes with a maximal Cauchy surface S. If S is compact and Conditions (1) - (3) of Theorem 1 hold, Then conditions (1) - (3) of Theorem 3 hold.

4. Totally Vicious Spacetimes

Definition: A spacetime (M,g) will be called totally vicious if $I^+(q) \cap I^-(q) = M$ for some point q in M. (Notice that if $I^+(q) \cap I^-(q) = M$ is true for one point q in M, it will be true for all points q in M; every point in M can be connected to every other point by both a future-directed and a past-directed timelike curve.)

The Goedel Universe, the Kerr-Newman solution with $a^2 + e^2 > m^2$ $(a \neq 0)$, and Minkowski space with the hyperplanes t=0 and t=1 identified are examples of totally vicious spacetimes. Totally vicious spacetimes are causally and chronologically symmetric about any point and any set in the spacetime. One property of such spacetimes is given by:

Proposition 3: A totally vicious spacetime is causally simple.

Proof: Recall that a spacetime is said to be causally simple if for every compact set K contained in M, $J^+(K)$ and $J^-(K)$ are closed. (HE, p.206). Let q be a point in K. Then we have $I^+(q) = M = I^+(K) = J^+(K)$, so $J^+(K)$ is closed. Similarly, $J^-(K)$ is closed. Hence, totally vicious spacetimes are causally simple.

This Proposition constitutes a counter-example to a theorem of Hawking and Sachs[14]: A causally simple spacetime is stably causal. The stable causality condition is said to hold on (M,g) if there are no closed timelike lines in both the metric g originally placed on M and in all metrics g' on M which are "near" g. (For a precise definition of "near" see Section 5) Clearly totally vicious spacetimes are not stably causal.

The proof of the Hawking-Sachs Theorem as given by those authors assumes that causally simple spacetimes are distinguishing (a spacetime is said to be distinguishing if for all points q and p, $I^-(q) = I^-(p)$ or $I^+(q) = I^+(p)$ implies q = p.) However, this condition is not included in the usual definition of causally simple, which is apparently the one used by Hawking and Sachs. Proposition 3 is really a defect in the usual definition of causally simple, for we have

<u>Proposition 4</u>: A spacetime (M,g) which contains closed timelike lines but which is not totally vicious is not causally simple.

<u>Proof</u>: Since (M,g) contains closed timelike lines, there is a point $q \in M$ for which $I^+(q) \cap I^-(q) \neq \phi$. Since (M,g) is not totally vicious, $\dot{J}^+(q) \cup \dot{J}^-(q)$ is non-empty. Suppose $\dot{J}^+(q) \neq \phi$ and let $p \in \dot{J}^+(q)$. If (M,g) were simply causal, then $\dot{J}^+(q) = E^+(q) = J^+(q) - I^+(q)$, so $q < p$, but not $q \ll p$. However, we have $q \ll q$ and $q < p$, which imply $q \ll p$. Similarly, we can deduce a contradiction between the assumptions $\dot{J}^-(q) \neq \phi$ and causal simplicity. Thus if (M,g) were causally simple, $\dot{J}^+(q) \cup \dot{J}^-(q)$ would have to be empty, and this is impossible.

5. Stability

As mentioned in the Introduction, there are two notions of stability in General Relativity. The first is the continued existence of a spacetime property under perturbations of the initial data. To be more precise,

Definition: A spacetime property will be said to be D - stable (for development stability) about a spacelike hypersurface S with initial data $(h^{\circ}_{ab}, \chi^{\circ}_{ab}, \Psi^{\circ}_{(i)})$ if the property exists in all spacetimes maximally developed from initial data $(h_{ab}, \chi_{ab}, \Psi_{(i)})$ in some neighborhood of the initial data $(h^{\circ}_{ab}, \chi^{\circ}_{ab}, \Psi^{\circ}_{(i)})$ on S in the original spacetime, where $\Psi_{(i)}$ denotes the other fields and their derivatives on S. We use the original metric h°_{ab} on S to define a distance function and hence a topology on the space of initial data on S. (i.e., we use the C^{∞} open topology on this space - see HE, p. 198 and reference 15 for more details). All sets of initial data are required to satisfy the constraint equations[16]. (All h_{ab} are required to be positive definite.)

We have:

Theorem 4: Singularity symmetry is a D - stable property of the initial data described in Theorem 1.

That is, given a compact maximal spacelike hypersurface S with $R_{ab}V^aV^b > 0$ everywhere, we can perturb the initial data slightly (so that S is no longer maximal, but still $|\chi^a{}_a| < \varepsilon$, for some $\varepsilon > 0$.) and still obtain incomplete timelike geodesics both to the past and to the future of S.

Proof: A change in the metric will change $R_{ab}V^aV^b$, but it still will be bounded away from zero on S for a change sufficiently small. Then there is an $\varepsilon > 0$ such that when $|\chi^a{}_a| < \varepsilon$, every timelike geodesic intersecting S orthogonally still has a point conjugate to S both to the past and to the future of S, provided every geodesic is both past and future complete. The existence of a geodesic which is incomplete to the future of S and one which is incomplete to the past of S then follows as in the proof of Theorem 2.

Similarly, we can show that singularity symmetry still occurs if we relax the maximal hypersurface condition of Theorem 3 to $|\chi^a{}_a| < \varepsilon$ on S for some $\varepsilon > 0$, the precise value of ε being determined by the constants a and b. However, it is not possible to prove D - stability with the initial data of Theorem 3 because we do not know if S would still be a Cauchy surface when the initial data is perturbed: it is not known if global hyperbolicity is a D - stable property about an S with the initial data of Theorem 3. It probably is not; an arbitrarily small amount of electric field added to Schwarzschild initial data can convert the resulting spacetime from Schwarzschild to Reissner-Nordström, and the former is globally hyperbolic while the latter is not.

The second notion of stability in General Relativity is the continued existence of a spacetime property under arbitrary, sufficiently small variations in the metric. To make this notion precise, we follow Geroch[17] and introduce a topology on the collection G of all Lorentz metrics on M. Let g'_{ab}, $\tilde{g}_{ab} \in G$. We will write $g'_{ab} < \tilde{g}_{ab}$ if every vector which is timelike or null with respect to g'_{ab} is timelike with respect to $\tilde{g}_{ab}$. That is, the light cones of $\tilde{g}_{ab}$ are "larger" than those of g'_{ab}. The set of $g_{ab} \in G$ such that $g'_{ab} < g_{ab} < \tilde{g}_{ab}$ forms a basis for a topology for G: the C^0 open topology (reference 15; HE, p. 198).

Definition: A property of spacetime is said to be G - stable (for global stability) if given any M, the collection of Lorentz metrics on M which have the given property forms an open set in G.

Theorem 5: Total viciousness is a G - stable property of spacetime.

Proof: Let (M,g) be a totally vicious spacetime. Clearly any metric $\tilde{g}_{ab}$ with $g_{ab} < \tilde{g}_{ab}$ is also totally vicious: if we "expand" the light cones at each point, then any closed timelike line in g_{ab} is also a closed timelike line in $\tilde{g}_{ab}$. To show that there exists a metric g'_{ab} with $g'_{ab} < g_{ab}$ for which (M,g') is totally vicious we proceed as follows. Suppose there is no such metric g'_{ab}. Then there exists a point $p \in M$ through which no closed timelike line passes for any $g'_{ab} < g_{ab}$. For if there were no such point p then the chronology violating sets would cover (M,g'_{ab}) for some $g'_{ab} < g_{ab}$. But the set of points at which the chronology condition is violated is the disjoint union of open sets of the form $I^+(q, g'_{ab}) \cap I^-(q, g'_{ab})$, $q \in M$ (HE, p. 189). Thus if these sets covered M they could not be disjoint unless they consisted only of one set; i.e., $I^+(q, g'_{ab}) \cap I^-(q, g'_{ab}) = M$.

However, the existence of such a point p is impossible, because given any closed timelike line γ (timelike in g_{ab}) through p, there is always a metric g'_{ab} with $g'_{ab} < g_{ab}$ for which γ is still everywhere timelike. (Given a timelike curve γ of finite time length, we can always "shrink" the light cones at all the points of γ such that γ is still everywhere timelike.)

We have a contradiction, and so there must exist a totally vicious spacetime (M,g') with $g'_{ab} < g_{ab}$.

Totally vicious spacetimes are in a real sense mirror images of stably causal spacetimes: both classes are G - stable, and both are defined by closed timelike lines - the former by their presence, and the latter by their absence.

References and Footnotes

[1] R.H. Gowdy, Phys. Rev. Lett. 27, 826 (1971).

[2] A. Einstein, The Meaning of Relativity, Fifth edition (Princeton University Press, Princeton, 1950), p. 107.

[3] C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation (Freeman and Company, San Francisco, 1973), p. 543,1181.

[4] It should be mentioned, however, that at present the experimental evidence is against closure. See J.R. Gott, J.E. Gunn, D.N. Schramm, and B.M. Tinsley, Ap. J. 194, 543 (1974).

[5] P.J.E. Peebles, Physical Cosmology (Princeton University Press, Princeton, 1971).

[6] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Spacetime (Cambridge University Press, Cambridge, 1973).

[7] Reference 3, p. 535.

[8] The first fruitful use of the concept of time symmetry was made by D.R. Brill in Ann. Phys. 7, 466 (1959), and by H. Araki in Ann. Phys. 7,456 (1959). These authors use the word "time symmetry" to mean the existence of a spacelike hypersurface S with $\chi_{ab} = 0$ and $D^{+}(S) \approx D^{-}(S)$. (For the precise definition see D.R. Brill, Time Symmetric Solutions of the Einstein Equations: Initial Value Problem and Positive Definite Mass, Ph.D. thesis, Princeton University, 1959).

[9] B.K. Harrison, K.S. Thorne, M. Wakano, and J.A. Wheeler, Gravitation Theory and Gravitational Collapse (University of Chicago Press, Chicago, 1965) p. 13.

[10] A maximal hypersurface is one for which the trace of the extrinsic curvature vanishes: $\chi^{a}_{\ a} = 0$.

11 D.R. Brill and F.J. Flaherty, preceeding paper; see also D.R. Brill and F.J. Flaherty, Commun. Math. Phys., to appear.

12 S.W. Hawking and R. Penrose, Proc. Roy. Soc. A 314, 529 (1970).

13 R. Penrose, Techniques of Differential Topology in Relativity, Vol. 7 of the Regional Conference Series in Applied Mathematics (SIAM, Philadelphia, 1972).

14 S.W. Hawking and R.K. Sachs, Commun. Math. Phys. 35, 287 (1974).

15 S.W. Hawking, Gen. Rel. and Grav. 1, 393 (1971).

16 D.R. Brill and S. Deser, Commun. Math. Phys. 32, 291 (1973).

17 R. Geroch, J. Math. Phys. 11, 437 (1970).

Biographical Sketch

I was born and raised in Andalusia, a small town in southern Alabama. My interest in physics dates back to my kindergarten days (<u>circa</u> 1952) when I became fascinated with von Braun's visions of interplanetary flight. By the time I entered M.I.T. as an 18 year old freshman in 1965, however, this interest had metamorphosed into interest in fundamental physics, with particular attention to the role of Time in scientific theories. Graduating from M.I.T. in 1969, I became a graduate student at the University of Maryland, where I am now working toward a Ph.D. in General Relativity with Dieter Brill as thesis supervisor.

My outside interests include hiking, reading Russian literature and science fiction, and studying history and philosophy.

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