Is the Universe Big Enough To Contain All Possibilities?

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ABSTRACT

Ideas from transfinite arithmetic are used to determine whether the set of all possible cosmological initial conditions can be physically realized in a single chaotic inflationary universe. It is shown that closed and flat inflationary universes cannot contain all such possibilities. Certain types of open inflationary universes can physically contain all such possibilities, but only at the expense of creating a "duplication problem". We show that in both standard quantum field theory and string field theory, there can be at most $2^{\aleph_0}$ physically distinct possible histories.

The most attractive feature of chaotic inflation is its promise to eliminate the problem of the boundary condition by claiming that all possible boundary conditions are physically realized in some "domain" somewhere in the universe at the present time. If this were true, it would allow us to account for certain unusual features we find in the Visible Universe, because all possibilities are realized somewhere. We shall give several definitions of "domain" that have been used in cosmology, and show that indeed "domains" so defined can have only a finite number of distinct physical states, a number which we calculate from very general physical principles. This result supports the philosophy of chaotic inflation. However, we go on to show that this result is misleading, because in many cosmological models, these domains do not remain forever causally separated. If the entire universe is, for example, flat Friedman-Roberson-Walker (FRW) in the large, or if it has the usual causal structure of many inflationary universes, then the domains will eventually coalesce into a single domain in the ultimate future. From the point of view of the ultimate future, there is only one single domain, but a continuum of possible initial data sets for this domain. Thus in these cases, the universe is not big enough to contain all possibilities. We give a general proof that there can be at most $2^{\aleph_0}$ physically different possibilities.

Even if the Universe were big enough — our argument fails for certain types of chaotic inflation universes — then we show that the realization of all boundary conditions causes what we term the Duplication Problem: it is overwhelmingly probable that all possibilities are realized not merely once, but realized an infinite number of times.

A *domain* is a spacelike region points of which were once in causal contact. That is, a domain is of the form $S \cap J^+(p)$, where S is a spacelike hypersurface and $J^+(p)$ is the causal future[1] of the point p, with p being either an event in space-time, or a point of the initial singularity (defined via Penrose's c-boundary construction[1]). For example, in a dust-dominated flat FRW universe, $S \cap J^+(p)$ will be a spherical ball of radius $3cH_0^{-1} \approx 3 \times (10^{10}$ lyrs) $\approx 3 \times (10^{28}$ cm) if p is point on the initial singularity, and S is the hypersurface of isotropy and homogeneity now. For closed or open dust-dominated FRW universes, $S \cap J^+(p)$ will be slightly smaller or larger than $3cH_0^{-1}$ respectively. In the inflationary universes, p is typically chosen to be an event at which the density is a few orders of magnitude less than the Planck density, S is locally FRW to a very high degree of approximation, and $S \cap J^+(p)$ is a spherical ball of radius $(L_P)\exp(2\pi M_P^2/m^2) \approx 10^3 \times 10^{14}$ lyrs, where $M_P \approx 10^{-5}$ gm and $L_P \approx 10^{-33}$ cm are the Planck mass and length respectively[2-5], and $m \approx 10^{-7} M_P$ (see ref. [2], p. 15) if inflation is generated by a scalar field ϕ with self-interaction potential $V(\phi) = m^2\phi^2/2$; the details change for other forms of $V(\phi)$ but the conclusions are similar.

In both flat or open FRW and inflationary space-times, the global space-time topology is $R^3 \times R^1$, where R^3 is a spacelike Cauchy hypersurface[1]. Thus the total cosmos at any one time is the union of a countable ($\aleph_0$) number of (possibly overlapping) domains. Each domain can be regarded as an independent mini-universe. How many such *physically distinct* mini-universes are there? Since the initial data of each domain is given by a finite number of real functions over such a subset of R^3, this is equivalent to asking how many physically distinct real functions can be defined over such a domain. For example, it is well-known that in general relativity there are two true degrees of freedom of the gravitational field at each space-time point; these two degrees of freedom at each point define two real C^2 functions making up initial data space for the empty space Einstein equations. Additional degrees of freedom are necessary to accommodate matter fields.

The set of all real functions on the real line (or on any finite dimensional manifold) is the power set of the continuum, $2^{\aleph_1}$, which is at least of size $\aleph_2$. (If the generalized continuum hypothesis is true, then $2^{\aleph_1} = \aleph_2$. Gödel and Cohen have shown that both the generalized continuum hypothesis and the continuum hypothesis are independent of standard set theory, based on the Zermelo-Frankel axioms.) However, the set of all real *continuous* functions on the real line or any finite dimensional manifold is only of cardinality $2^{\aleph_0}$ (which equals $\aleph_1$ if the Continuum Hypothesis is true)[6]. This theorem results from the fact that continuous functions are defined by the values they assume at rational values of the independent variable. (Similarly, C^n functions for $1 \leq n < \infty$, are also of cardinality $2^{\aleph_0}$).

To see this, note first that the cardinality of the continuous functions from $\Re \to \Re$ cannot exceed that of all functions, $2^{\aleph_1}$; and cannot be less than $2^{\aleph_0}$, the number of *constant* continuous functions. If we consider the continuous functions restricted to the rationals and pick two continuous functions, f and g, then f - g must be non-zero somewhere and hence, by continuity, must be non-zero in some open neighbourhood as well. Since there must be some rational in that neighbourhood, the restriction of f to the rationals cannot be equal to the restriction of g to the rationals. Hence, the mapping from the set of continuous functions $\Re \to \Re$ into the set of functions from the rationals to the reals is one-to-one and so the cardinality of the continuous functions cannot exceed that of the functions from the rationals to the reals, which is $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}$. But since we showed this cardinality cannot be less than $2^{\aleph_0}$, it must be equal to it.

This argument can be generalized to show that the cardinality of continuous functionals on the space of continuous functions from $\Re \to \Re$ is also $2^{\aleph_0}$. Further, the cardinality of continuous functionals on the space of continuous functionals on the space of continuous functions is $2^{\aleph_0}$, and so on. In contrast, if the continuity requirement is not imposed, the cardinality of *all* functionals of *all* functions from $\Re \rightarrow \Re$ is the power set of $2^{\aleph_1}$, which is at least of size $\aleph_3$, and the cardinality of *all* functionals of *all* functionals of *all* functions from $\Re \rightarrow \Re$ is the power set of the power set of $2^{\aleph_1}$, which is at least of size $\aleph_4$. With the continuity requirement imposed at each level, the cardinality of possibilities never rises above $2^{\aleph_0}$. (We assume here the standard function and functional space topologies.)

This is extremely important physically, because as Page[7] has pointed out, the quantization of a classical field theory means introducing a new physical quantity, the probability amplitude of the classical field, which is a functional of the classical field. Thus, if the probability amplitude of the field is required to be a continuous functional of the field — this is the standard requirement of quantum field theory — then the cardinality of physically distinct possibilities is not increased. In the language of the Griffiths-Omnes-Gell-Mann/Hartle Many-Histories interpretation of quantum mechanics, this means that there are only $2^{\aleph_0}$ different histories.

In string field theory, the physical quantity is a probability amplitude for each string field, and each string field component is a functional of the functions which give the spatial positions along the string loop. In other words, the probability amplitude in quantum string field theory is a functional of a functional of functions from $\Re \rightarrow \Re$. As in quantum field theory, continuity is imposed at each stage, so even in string field theory there are only $2^{\aleph_0}$ different histories.

Whether the continuum hypothesis is true or not, it remains true that the cardinality of the number of possible initial data sets on a domain is greater than the cardinality of the number of domains. Thus, using cardinality as the measure of the relative number of domains, we conclude that the number of realized domains is of measure zero in space of possible universes. If each distinct continuous function is to be thought of as a physically distinct initial data set, then the universe is not big enough to contain all possibilities.

Cardinality is a coarse measure of the size of the number of possible domains. One might hope to use a finer measure, but Ulam[8] has established that no *universal* measure exists on the set of all continuous functions of a finite dimensional manifold, where a universe measure μ is one satisfying 4 conditions: (1) if A and B are disjoint subsets of X, the $\mu(A \cup B) = \mu(A) + \mu(B)$; (2) $\mu(A) = 0$ whenever A consists of just one element of X; (3) $\mu(X) = 1$; and (4) $\mu(A)$ is defined for *all* subsets A of X. Thus cardinality seems the best measure we are likely to get. Furthermore, the standard norm one puts on initial data space in general relativity is the Sobolev norm[1], in which two functions f and g are distinct if $\int |f - g|^2 dx$ is non-zero, which it will be if f and g are continuous and not identical. There are indeed at least $\aleph_1$ truly distinct "points" in the standard initial data space of the empty space Einstein equations; the same is true if we add the standard matter fields[1].

Of course, being mathematically distinct does not necessarily mean physically distinct. In fact, we would expect quantum mechanics to discretize initial data space. Such a discretization is what causes all non-relativistic quantum mechanical systems to be non-chaotic[9-11] with respect to the evolution of the wave function (actually, such systems are almost periodic), whereas most classical systems are chaotic[12,13]. In classical systems with n degrees of freedom, there is an infinite amount of structure no matter finely one divides the phase space, whereas in the corresponding quantum systems, two systems within h^{3n} of each other in phase space are physically equivalent. Classically, the $\aleph_1$ mathematically distinct points of the initial data space of general relativity are indeed physically distinct, but quantum mechanics makes the number at most a countable infinity.

The number of physically distinguishable domains is actually finite at the present time. This number (of domains which could be causally connected at the present universal epoch) is obtained by using the Bekenstein bound[14,15] on the number of quantum states that can be inside a 2-sphere of proper radius R inside which the total non-gravitational energy is E: (number of states) $\leq \exp[(2\pi)^2 ER/hc] = \exp[16\pi^3\rho R^4/3hc] = \exp[9\pi(3cH_0^{-1}/L_P)^2] = \exp[10^{124}]$ where ρ is the average energy density in the 2-sphere, and we have used $\rho = 3(cH_0)^2/8\pi G$for flat FRW. This number $\exp[10^{124}] \approx 10^{10^{124}}$ is the Penrose number ([16], p. 344), the number of quantum states in the visible universe (Penrose uses for cH_0^{-1} the size of the visible universe rather than $3cH_0^{-1}$, so he gets $10^{10^{123}}$ rather than $10^{10^{124}}$). The domain size in Linde's chaotic inflation with quadratic potential which we are using as an example is $R = (L_P)\exp(2\pi M_P^2/m^2)$, while the density is that of flat FRW. Hence $\exp[16\pi^3\rho R^4/3hc] = \exp[16\pi^3(3H_0^2c^2/8\pi G)(L_P\exp(2\pi M_P^2/m^2))^4] = \exp[\pi\{L_P/(cH_0^{-1})\}^2\exp(8\pi M_P^2/m^2)] \approx \exp[\exp(8\pi M_P^2/m^2)] = \exp[10^{10^{15}}]$, which we shall term the "Linde number." Depending on the physics which operates at early times, the number of physically distinct causally connected domains now is either the Penrose number or the Linde number. Thus it might appear that all physically distinct possibilities can and would be realized, since there would be an infinite number ($\aleph_0$) of domains in the universe at the present time but only a finite number of physically distinct ones.

However, this is not the true state of affairs because if the residual cosmological constant is zero, then the domains do not remain forever out of causal contact. Eventually they will come into causal contact, forming "super-domains", and we can show that the cardinality of physically distinguishable super-domains is $2^{\aleph_0}$. Since the number of physically realizable super-domains is only of cardinality $\aleph_0$, this will mean that almost all physically possible super-domains are never physically realized.

Let us first consider the open and flat FRW universes. Cover a spacelike slice by the usual (χ,θ,ϕ) coordinates, where the radial coordinate ranges over $0 < \chi \leq +\infty$. Consider the domains along a single line in the χ direction by fixing θ and ϕ. As χ goes to infinity, an infinite number of domains will be swept out, each domain having an average proper radial size $g_{\chi\chi}\Delta\chi_0 = R(t)\Delta\chi_0$, where $\Delta\chi_0$ is a constant. The number of domains at a given time t at a coordinate distance χ from the origin increases as χ^2 in the flat case, and as $\sinh^2\chi \approx (1/4)\exp 2\chi$ for large χ in the open case.

A sufficient condition for two domains to be considered physically part of a single super-domain is that it must be possible to send a signal from one domain to the other which can be recorded permanently in the second domain. An early universe version of this requirement is the defining characteristic of the domains: a domain is a spatial region which was once in significant causal contact. So we are simply extending the definition of domain into the future. Since there are no event horizons for the comoving observers in the open or flat FRW universes, light rays can be sent back and forth between any two domains no matter how distant in coordinate χ an infinite number of times. However, because of the redshift, the absence of horizons is not sufficient to guarantee that a signal can be sent between domains an arbitrary distance apart.

If the domains are labeled 0,1,2, ... outward from the selected central domain (labeled 0), and if the signal energy emitted from domain N is E_0, while E_N is the signal energy of the signal from domain N measured in the central domain 0, then $E_N/E_0 = R(t_N)/R(t_0)$, where t_N is the time of receipt and t_N is the time of emission. In both the flat and open FRW universes the scale factor can be approximated by $R(t) = R_0 t^m$, for m some constant. Thus $E_N/E_0 = (t_0/t_N)^m$. Furthermore, if χ_N and χ_0 are respectively the radial coordinate of the Nth domain and the central domain, the equation for light rays $ds^2 = 0 = -dt^2 + R^2(t)d\chi^2$ implies $\chi_N - \chi_0 = R_0^{-1}\ln(t_N/t_0) = R_0^{-1}\ln(E_0/E_N)$ for $m = 1$, and $\chi_N - \chi_0 = ([1-m]R_0)^{-1}(t_N^{1-m} - t_0^{1-m}) =$
$([1-m]R_0)^{-1}t_0^{1-m}\{(E_0/E_N)^{1/m-1} - 1\}$ for $m \neq 1$. Since $\chi_N - \chi_0 = N\Delta\chi_0$, we have $E_N \approx (\text{constant})E_0\exp(-N)$ and $t_N \propto \exp(N)$ for $m = 1$. We also have $E_N \approx (\text{constant})E_0 N^{m/(m-1)}$ and $t_N \propto N^{1/(1-m)}$ for $m \neq 1$ and large t ($t >> t_0$). In particular, for dust dominated flat universes we have $E_N/E_0 \propto N^{-2}$ and $t_N \propto N^3$, for large N. For radiation dominated flat universes we have $E_N/E_0 \propto N^{-1}$ and $t_N \propto N^2$, for large N. Only if this photon of energy E_N from domain N can be detected in the central domain can domain N be considered physically connected to the central domain. We

have shown[17] that in the far future, protons and hence heavier nuclei will decay by spontaneous formation and evaporation of mini-black holes inside such hadrons within 10^{122} years. After that, one way to record the receipt of a photon is by a state change in positronium. (Alternatively, if R-parity is conserved, one could appeal to the analogous particle-antiparticle bound state formed by the lightest supersymmetric particle unless it is a Majorana particle.) For illustration we shall just consider the positronium state since it will be more tightly bound. The signal photon from domain N has sufficient energy to induce a state change in one positronium atom in the central domain for any N. The energy ΔE required to induce a transition between the lower energy state with electron and positron spins antiparallel and the higher energy parallel state decreases as $\Delta E \propto n^{-3}$, where n is the principle quantum number of positronium[17]. Further, the lifetime of the parallel state increases as $n^{9/2}$. Mechanisms[17] can be given to ensure that the given positronium atom in the central domain is in the Nth state when the photon is received in the central domain. Thus provided the positronium atom in the central domain is in the n = N state when the photon from the Nth domain is received, the signal photon will be sufficiently energetic to induce a transition in both the radiation and dust-dominated flat universes. Because of the exponential damping of the photon energy and the exponential growth of the time of receipt in the open universe, there is a maximal distance from the central domain beyond which a domain cannot reasonably be considered physically a part of the central domain.

The global causal structure of space-time with eternal chaotic inflation with zero cosmological constant is the same as Minkowski space: in the large, eternal inflation is temporally static, with the domains being, in effect, little closed universes which eventually recollapse into the vacuum. The overall space-time topology is R^4, and in the large homogeneous and isotropic, with zero mean energy density. There are no event horizons for the fundamental observers in such a space, and provided signals can be transmitted across the walls separating the domains, it is possible for signals from any domain N to pass to the central domain, as there is no overall redshift to be overcome.

However, the "space-time" of eternal chaotic inflation might have a fractal structure, with baby universes budding off baby universes and so on ad infinitum. If this is the overall structure, then "space-time" is not a differential manifold, and hence not a space-time. But Linde has shown[2,14,15] that in this case, it is possible to send a signal from the parent universe to any of the subsequent baby universes if the cosmological constant is positive but less than 10^{-10^7}. This fact combined with the argument in the preceding paragraph means that in eternal chaotic inflation, signals can be sent from any domain to any other, and hence from the point of view of the ultimate future, all domains must be regarded as a single domain. Consider again the ordering of the $\aleph_0$

domains 0,1,2,... . Since as we showed, each domain is bounded above in its number S_0 of possible states, the number of physically distinguishable possible states of a super-domain is $S_0 \times 2^{\aleph_0} = 2^{\aleph_0}$. Since, ultimately, only one of these can be realized, the Universe is not big enough to contain all physical possibilities.

There may be one way of avoiding this conclusion. If we regard the entire universe not as an infinite sequence of buddings of baby universes from a single initial baby universe, but rather as an infinite causally disjoint collection of such sequences — this could happen if we imagine not a single "initial" baby universe, but an infinite number of "initial" baby universes — then there could be $2^{\aleph_0}$ such sequences *if* the "initial" baby universes were uncountable. ("Initial" is in quotes here because there is no way to define a global time coordinate on the space of such causally disjoint universes sequences.) In this scenario, the cardinality of such sequences would equal the cardinality of possibilities, and the universe could indeed be big enough to contain all physical possibilities. Such a cosmology with an infinite number of "initial" universes has never been developed in the literature, but it is required if the Universe is to be big enough to contain all physical possibilities. One possible way of generating such a cosmology is to simply postulate that the "initial" universes already exhaust all the $2^{\aleph_0}$ possibilities.

If there actually are an infinite number of domains, and if the number of physically distinguishable states in a domain is finite, then a strange implication is that, with probability one, every person and every person's every action is repeated an infinite number of times. This we shall call the "Duplication Problem." Most of the major cosmological models — the steady state universe, the open and flat FRW universes, an oscillating closed FRW universe, chaotic inflationary cosmology, and the Hartle-Hawking quantum universe — suffer from this problem[18]. The only major model that can avoid the problem is a closed universe which starts in a Big Bang and ends in a Big Crunch[19,20]. And even in this model the Duplication Problem is only avoided if the universe is small enough. For example, if (1) the universe is infinite in spatial extent; if (2) the universe is approximately homogeneous; (3) if an Earthlike planet occurs at least once per Hubble volume; and (4) if each possible gene collection up to the human complexity is equally likely, then there exists now an infinite number of genetic clones of each reader of this article, and each such clone is on the average of 10^{10^7} lyrs apart[1]. An upper bound[21] for the amount of information that can be coded in the human brain is 10^{15} bits, so each reader of this article will be duplicated in both genome and personality in every volume of radius $10^{10^{14}}$ lyrs. In chaotic inflationary cosmology generated by a massive scalar field, the typical domain has a radius of about $10^{3 \times 10^{14}}$ lyrs, and hence each reader of this article is with high probability duplicated in both genome and personality many times in our own domain. This "Duplication Problem" is a consequence of chaotic inflation and indeed of virtually any cosmological model that resolves the Flatness Problem by making the universe huge. Indeed, with high probability, any universe big enough to contain all possibilities must also be big enough to contain all possibilities an infinite number of times. This limitless duplication of each and every one of us is certainly unattractive. But of one thing regarding this conclusion we can be certain: if true, it certainly cannot be original!

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